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Theory of Multiplicative Metric Space: An Overview

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In a broad range of mathematical problems, the existence of solution is equivalent to existence of a fixed point for a suitable transformation. The existence of fixed-point theory is therefore having a tremendous importance and beauty in several areas of mathematics and other sciences. It is possible that a problem does not have any of the solution but the fixed-point theory itself provides the condition under which a transformation has solutions. This is the theory which is embellished mixture of pure and applied analysis, topology and geometry.

It is fruitful and promising area of research for mathematicians during the last several decades. Over since last 70 years, fixed point theory has been revealed itself as a very powerful and important tool in the study of non-linear phenomena. In particular, fixed-point techniques have been applied in diverse fields such as in biology, chemistry, economics, engineering, game theory and physics. The point at which the curve y = f (x) and the line y = x intersects gives the solution of the curve, and the point of intersection is the fixed point of the curve. The usefulness of the concrete applications has increased enormously due to the development of accurate techniques for computing fixed points.

Fixed points are the points which remain invariant under a map/transformation. Fixed points tell us which parts of the space are pinned in plane, not moved, by the transformation. The fixed points of a transformation restrict the motion of the space under some restrictions.

We note that fixed point problems and root finding problems f(x) = 0 are equivalent.

Now, the question arise what type of problems have the fixed point. The fixed point problems can be elaborated in the following manner:

(i) What functions/maps have a fixed point?

(ii) How do we determine the fixed point?

(iii) Is the fixed point unique?

Next, we state a result which gives us the guarantee of existence of fixed points.

Suppose g is continuous self map on [a, b]. Then, we have the following conclusions:

(i) If the range of the mapping y = g(x) satisfies y ∈ [a, b] for all x ∈ [a, b], then g has a fixed point in [a, b].

(ii) Suppose that g'(x) is defined over (a, b) and that a positive constant k < 1 exists with |g'(x)| ≤ k for all x ∈ (a, b), then g has a unique fixed point in [a, b].

Now, suppose that (X, d) be a complete metric space and T : X → X be a map. The mapping T satisfies a Lipchitz condition with constant $\alpha \geq 0$ such that d (Tx, Ty) ≤ α d(x, y), for all x, y in X. For different values of α, we have the following cases:

(a) T is called a **contraction** mapping if $\alpha < 1$;

(b) T is called **non-expansive** if $\alpha \leq 1$;

(c) T is called **contractive** if $\alpha = 1$.

It is clear that contraction ⇒ contractive ⇒ non-expansive⇒ Lipschitz. However, converse may not true in either case as:

(i) The identity map I : X → X, where X is a metric space, is non-expansive but not contractive.

(ii) Let X = [0, ∞) be a complete metric space equipped with the metric of absolute value. Define, f : X →X given by f(x) = x + 1/x. Then f is contractive map, while f is not a contraction.

Multiplicative metric space

In 1991, Muttalip Ozavsar and Adem Cevikal discussed multiplicative mappings by giving some topological properties of the relevant multiplicative metric space. It was observed that the set of positive real numbers is a complete multiplicative metric space with respect to the multiplicative absolute value function. Also some concepts of multiplicative contraction mapping had used and some fixed point theorems were proved of such mappings on a complete multiplicative metric space. Muttalip Ozavsar and Adem Cevikal states

Let X be a non-empty set. Suppose that the mapping

$d : X \times X \to \mathbb{R}^+$ satisfying the following conditions:

(M_1) $d(x, y) \geq 1$ for all $x, y \in X$ and $d(x, y) = 1$ iff $x = y$,

(M_2) $d(x, y) = d(y, x)$ for all $x, y \in X$,

(M_3) $d(x, z) \leq d(x, y)\cdot d(y, z)$ for all $x, y, z \in X$ (multiplicative triangle inequality).

Definition 1.4 Let (X, d) be a multiplicative metric space, $x \in X$ and $\varepsilon > 1$. We define a set $B_\varepsilon(x) = \{y \in X \mid d(x, y) < \varepsilon\}$,which is called **multiplicative open ball** of radius ε with centre x.

Similarly, one can describe multiplicative closed ball as $B_\varepsilon(x) = \{y \in X \mid d(x, y) \leq \varepsilon\}$.

Definition 1.5 Let (X, d) be a multiplicative metric space and $A \subset X$. Then we call $x \in A$ a multiplicative interior point of A if there exists an $\varepsilon > 1$ such that $B_\varepsilon(x) \subset A$. The collection of all interior points of A is called **multiplicative interior of A** and denoted by int (A).

Definition 1.6 Let (X, d) be a multiplicative metric space and $A \subset X$. If every point of A is a multiplicative interior point of A, i.e., A = int (A), then A is called a **multiplicative open set**.

Lemma 1.7 Let (X, d) be a multiplicative metric space. Each multiplicative open ball of X is a multiplicative open set.

Proof. Let $x \in X$ and $B_\varepsilon(x)$ be a multiplicative open ball. For $y \in B_\varepsilon(x)$, if we let $\delta = \frac{\varepsilon}{d(x,y)}$ and $z \in B_\delta(y)$, then $d(y, z) < \frac{\varepsilon}{d(x,y)}$, from which we conclude that

$d(x, z) < d(x, y) \cdot d(y, z) < \varepsilon$.

This shows that $z \in B_\varepsilon(x)$, which means that $B_\delta(y) \subset B_\varepsilon(x)$ i.e., y is interior point of $B_\varepsilon(x)$.

Thus $B_\varepsilon(x)$ is multiplicative open set.

Lemma 1.8 The intersection of any finite family of multiplicative open sets is also a multiplicative open set.

Proof. Let B_1 and B_2 be two multiplicative open sets and $y \in B_1 \cap B_2$. Then there are $\delta_1, \delta_2 > 1$ such that $B_{\delta_1}(y) \subset B_1$ and $B_{\delta_2}(y) \subset B_2$. Letting δ be the smaller of δ_1 and δ_2, we conclude that $B_\delta(y) \subset B_1 \cap B_2$. Hence the intersection of any finite family of multiplicative open sets is a multiplicative open set.

Definition 1.9 Let (X, d) be a multiplicative metric space. A point $x \in X$ is said to be **multiplicative limit point** of $S \subset X$ if and only if $(B_\varepsilon(x) - \{x\}) \cap S \neq \phi$ for every $\varepsilon > 1$. The set of all multiplicative limit points of the set S is denoted by S'.

Definition 1.10 Let (X, d) be a multiplicative metric space. We call a set $S \subset X$ **multiplicative closed** in (X, d) if S contains all of its multiplicative limit points.

Definition 1.11 Let (X, d) be a multiplicative metric space, $\{x_n\}$ be a sequence in X and $x \in X$. If for every multiplicative open ball $B_\varepsilon(x)$, there exists a natural number N such that $n \geq N \Rightarrow x_n \in B_\varepsilon(x)$, then the sequence $\{x_n\}$ is said to be **multiplicative convergent** to x, denoted by $x_n \to x$ $(n \to \infty)$.

Lemma 1.12 Let (X, d) be a multiplicative metric space, $\{x_n\}$ be a sequence in X and $x \in X$. Then $x_n \to x$ $(n \to \infty)$ if and only if $d(x_n, x) \to 1$ $(n \to \infty)$.

Proof. Suppose that the sequence $\{x_n\}$ is multiplicative convergent to x. i.e., for every $\varepsilon > 1$, there is a natural number N such that $d(x_n, x) < \varepsilon$ whenever $n \geq N$. Thus we have the following inequality $1/\varepsilon < d(x_n, x) < 1\cdot\varepsilon$ for all $n \geq N$.

This means $|d(x_n, x)| < \varepsilon$ for all $n \geq N$, which implies that the sequence $d(x_n, x)$ is multiplicative convergent to 1. It is clear to verify the converse.

Lemma 1.13 Let (X, d) be a multiplicative metric space, $\{x_n\}$ be a sequence in X. If the sequence $\{x_n\}$ is multiplicative convergent, then the multiplicative limit point is unique.

Proof. Let $x, y \in X$ such that $x_n \to x$ and $x_n \to y$ $(n \to \infty)$. That is, for every $\varepsilon > 1$, there exists N such that, for all $n \geq N$, we have $d(x_n, x) < \sqrt{\varepsilon}$ and $d(x_n, y) < \sqrt{\varepsilon}$. Then, we have

$d(x, y) \leq d(x_n, x) \cdot d(x_n, y) < \varepsilon$. Since ε is arbitrary, $d(x, y) = 1$. This says $x = y$.

Theorem 1.14 Let (X, d_X) and (Y, d_Y) be two multiplicative metric spaces, $f : X \to Y$ be a mapping and $\{x_n\}$ be any sequence in X. Then f is multiplicative continuous at the point $x \in X$ iff $f(x_n) \to f(x)$ for every sequence $\{x_n\}$ with $x_n \to x$ $(n \to \infty)$.

Proof. Suppose that f is multiplicative continuous at the point x and $x_n \to x$. From the multiplicative continuity of f, we have that, for every ε > 1, there exists δ > 1 such that

$f(B_\delta(x)) \subset B_\varepsilon(f(x))$. Since $x_n \to x$ ($n \to \infty$), there exists N such that $n \geq N$ implies $x_n \in B_\delta(x)$. By virtue of the above inclusion, then $f(x_n) \in B_\varepsilon(f(x))$ and hence $f(x_n) \to f(x)$ ($n \to \infty$).

Conversely, assume that f is not multiplicative continuous at x. That is, there exists an ε > 1 such that, for each δ > 1, we have $x' \in X$ with $d_X(x', x) < \delta$ but

(1.2) $d_Y(f(x'), f(x)) \geq \varepsilon$

Now, take any sequence of real numbers (δ_n) such that $\delta_n \to 1$ and $\delta_n > 1$ for each n. For each n, select x' that satisfies the equation (1.2) and call this x_n'. It is clear that $x_n' \to x$, but $f(x_n')$ is not multiplicative convergent to f(x). Hence we see that if f is not multiplicative continuous, then not every sequence $\{x_n\}$ with $x_n \to x$ will yield a sequence $f(x_n) \to f(x)$. Taking the contrapositive of this statement demonstrates that the condition is sufficient.

Similarly, we can prove the following theorems.

Theorem 1.15 Let (X, d) be a multiplicative metric space and $\{x_n\}$ be a sequence in X. The sequence is multiplicative convergent, then it is a multiplicative cauchy sequence.

Proof. Let $x \in X$ such that $x_n \to x$. Hence we have that for any ε > 1, there exist a natural number N such that $d(x_n, x) < \sqrt{\varepsilon}$ and $d(x_m, x) < \sqrt{\varepsilon}$ for all $m, n \geq N$.

By the multiplicative triangle inequality, we get

$d(x_n, x_m) \leq d(x_n, x) \cdot d(x, x_m) < \sqrt{\varepsilon} \cdot \sqrt{\varepsilon} = \varepsilon$,

which implies $\{x_n\}$ is a multiplicative cauchy sequence.

Theorem 1.16 (Multiplicative characterization of supremum) Let A be a non-empty subset of $\mathbb{R}^+$. Then s = sup A if and only if

(i) $a \leq s$ for all $a \in A$

(ii) there exists at least a point $a \in A$ such that $|s / a| < \varepsilon$ for all ε > 1.

Proof. Let s = sup A. Then from the definition of supremum, the condition (i) is trivial. To prove the condition (i), assume that there is an ε >1 such that there are no elements $a \in A$ such that $|s / a| < \varepsilon$. If this is the case, then s / ε is also an upper bound for the set A. But this is impossible, since s is the smallest upper bound for A.

To prove the converse, assume that the number s satisfies the conditions (i) and (ii).

By the condition (i), s is an upper bound for the set A. If s ≠ sup A, then s > sup A and ε = s / sup A > 1.

By the condition (ii), there exists at least a number $a \in A$ such that $|s / a| < \varepsilon$. By the definition of the number ε, this means that a > sup A. This is impossible, hence s = sup A.

Theorem 1.17] Let $\{x_n\}$ and $\{y_n\}$ be multiplicative cauchy sequences in a multiplicative metric space (X,d). The sequence $\{d(x_n,y_n)\}$ is also a multiplicative cauchy sequence in the multiplicative metric space $(\mathbb{R}^+, d^*)$.

Lemma 1.12 Let (X, d) be a multiplicative metric space, $\{x_n\}$ be a sequence in X and $x \in X$. Then $x_n \to x$ ($n \to \infty$) if and only if $d(x_n, x) \to 1$ ($n \to \infty$).

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By the multiplicative triangle inequality, we get $d(x_n, x_m) \leq d(x_n, x) \cdot d(x, x_m) < \sqrt{\varepsilon} \cdot \sqrt{\varepsilon} = \varepsilon$, which implies $\{x_n\}$ is a multiplicative cauchy sequence.

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